

# Cyclotomic $p$ -adic Multi-Zeta Values in Depth Two

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**Abstract.** In this paper we compute the values of the  $p$ -adic multiple polylogarithms of depth two at roots of unity. Our method is to solve the fundamental differential equation satisfied by the crystalline Frobenius morphism using rigid analytic methods. The main result could be thought of as a computation in the  $p$ -adic theory of higher cyclotomy. We expect the result to be useful in proving non-vanishing results since it gives quite explicit formulas.

## 1. INTRODUCTION

Let  $\mathbb{G}_m := \text{Spec } \mathbb{Q}[z, z^{-1}]$  denote the multiplicative group and  $\mu_M$ , the kernel of multiplication by  $M$  on  $\mathbb{G}_m$ , for  $M \geq 1$ . Let  $Y_M := \mathbb{G}_m \setminus \mu_M$ , and  $Y'_M$  denote the base change of  $Y_M$  to  $L := \mathbb{Q}(\zeta)$ , where  $\zeta$  is a primitive  $M$ -th root of unity. Choosing appropriate (tangential) basepoints Deligne and Goncharov define the unipotent motivic fundamental group  $\pi_1^{\text{mot}}(Y'_M, \cdot)$  of  $Y'_M$  whose ring of functions is an ind-object in the tannakian category of mixed Tate motives over  $\mathcal{O}_L[M^{-1}]$  [4, §5].

The comparison between the Betti and de Rham realization of this fundamental group is completely described by the cyclotomic versions of multi-zeta values [4, Proposition 5.17]. Namely, fix an imbedding of  $L$  in  $\mathbb{C}$ , and identify  $L$  with its image. The Lie algebra of the de Rham fundamental group of  $Y'_M$  over  $\mathbb{C}$  is the free pro-nilpotent Lie algebra with generators  $\{e_i\}_{0 \leq i \leq M}$ , where  $e_0$  (resp.  $e_i$ ) correspond to taking residues at 0 (resp.  $\zeta^i$ ) (cf. §2.1.3 below). Hence the  $\mathbb{C}$ -valued points of the de Rham fundamental group is the set of group-like elements in the (non-commutative) formal power series ring  $\mathbb{C}\langle\langle e_0, \dots, e_M \rangle\rangle$  (loc. cit.). The image of the Betti path from the tangential basepoint 1 at 0 to the tangential basepoint -1 at 1 under the de Rham-Betti comparison isomorphism then gives a group-like element  ${}_1\gamma_0$  in  $\mathbb{C}\langle\langle e_0, \dots, e_M \rangle\rangle$ . Then by [4, Proposition 5.17], the coefficient of  $e_0^{s_m-1} e_{i_m} \dots e_0^{s_1-1} e_{i_1}$  in  ${}_1\gamma_0$ , where  $M \geq i_m, \dots, i_1 \geq 1$  and  $s_m > 1$ , is

$$(-1)^m \sum_{n_m > \dots > n_1 > 0} \frac{\zeta^{i_m(n_{m-1}-n_m) + \dots + (-i_1)n_1}}{n_m^{s_m} \dots n_1^{s_1}}.$$

The study of these numbers is the Hodge-theoretic analog of higher cyclotomy [6].

The main result below is the crystalline analog of the above for  $m \leq 2$ . We describe this in more detail. Letting  $p$  be a prime which does not divide  $M$ ,  $\pi_1^{\text{mot}}(Y'_M, \cdot)$  has good reduction modulo  $p$ , and hence one would expect a crystalline realization of this motive at  $p$ . This is completely described by the Frobenius action on the de Rham fundamental group  $\pi_{1,dR}(X'_M, \cdot)$ , where  $X'_M$  is the base change of  $Y'_M$  to  $\mathbb{Q}_p(\zeta)$ . Let  $g_i$  denote the image of the canonical de Rham path from the tangential

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basepoint 1 at 0 to the tangential basepoint 1 at  $\zeta^{i/p}$  (c.f. §2.2.3). As above  $g_i$  is naturally in  $\mathbb{Q}_p(\zeta)\langle e_0, \dots, e_M \rangle$ . The main result of the paper, Theorem 6.4.3 below, gives an explicit formula for the coefficient of  $e_j e_0^{s-1} e_k e_0^{t-1}$  in  $g_i$ , in terms of iterated sums, exactly as above. Since  $g_i$  is group-like, this also determines the coefficients of terms of the form  $e_0^{r-1} e_j e_0^{s-1} e_k e_0^{t-1}$ . This might be thought of as the  $p$ -adic theory of higher cyclotomy in depth two.

Next we describe the contents of the paper. In §2, we review the de Rham and crystalline fundamental groups of a curve in a way which will be suitable for our purposes. In particular, using the horizontality of the Frobenius with respect to the canonical connection we arrive at the fundamental differential equation (2.2.8). At the end of this section, we fix the notation for what follows. In §3, we obtain a certain relation between the coefficients of the power series expansions of rigid analytic functions on  $\mathcal{U}_M$ , which is essential for the computations (Corollary 3.0.4). In §4, we compute of the polylogarithmic part which is fairly straightforward. Next there is a section on the type of iterated sums that appear in the computations. These functions will appear as coefficients of the power series expansions above and will satisfy the hypotheses of Corollary 3.0.4, so the inductive process will continue. In §6, we will proceed with the computation, and finish with the main result in Theorem 6.4.3.

## 2. THE FUNDAMENTAL DIFFERENTIAL EQUATION

Fix a prime  $p$ , which does not divide  $M$ . Let  $X_M$  and  $X'_M$  denote the base change of  $Y_M$  to  $\mathbb{Q}_p$  and  $\mathbb{Q}_p(\zeta)$ . Let  $A_M$  and  $A'_M$  denote the rings of regular functions on  $X_M$  and  $X'_M$  respectively. Finally, let  $D_M := \overline{X'_M} \setminus X'_M$ .

**2.1. The de Rham fundamental group of  $X'_M$ .** We review the theory of the de Rham fundamental group [2, 10.24-10.53, §12], [8, §4, §5] in the case of  $X'_M$ .

**2.1.1. The fundamental torsor.** Let  $K$  be any field of characteristic 0 and  $X/K$  be a smooth and geometrically connected curve and  $\text{Mic}_{uni}(X/K)$  denote the category of vector bundles with integrable connection which are *unipotent*, i.e. that have a finite separated and exhaustive filtration by sub-bundles with connection such that the non-zero graded pieces are the trivial line bundle with connection. This category naturally forms a tensor category over  $K$  in the sense of [2, §5.2], [3].

Let  $S/K$  be a scheme over  $K$  and let  $\text{Vec}_S$  denote the category of locally free sheaves of finite rank on  $S$  and  $\omega : \text{Mic}_{uni}(X/K) \rightarrow \text{Vec}_S$  be a fiber functor [2, §5.9]. Then to  $\omega$  there is associated a  $K$ -groupoid acting over  $S$  [3, §1.6] called the *fundamental groupoid* of  $X$  at  $\omega$  and denoted by  $\mathcal{P}_{dR}(X, \omega)$ . The fundamental groupoid is faithfully flat and affine over  $S \times_K S$  and represents the functor on the category of  $S \times_K S$ -schemes whose  $T$ -valued points for any  $\pi : T \rightarrow S \times_K S$  is the set of  $\otimes$ -isomorphisms from  $\pi^* p_2^* \omega$  to  $\pi^* p_1^* \omega$ , where  $p_1, p_2 : S \times_K S \rightarrow S$  are the projections [3, §1.11, Théorème 1.12].

Taking the cartesian product of  $\mathcal{P}_{dR}(X, \omega) \rightarrow S \times_K S$  with the diagonal  $\Delta : S \rightarrow S \times_K S$  gives  $\pi_{1,dR}(X, \omega)$ , the *fundamental group* of  $X$  at the fiber functor  $\omega$ .

Let  $x \in S(K)$  then attaching  $\mathcal{F}(x)$ , the fiber of  $\mathcal{F}$  at  $x$ , to  $\mathcal{F} \in \text{Vec}_S$  gives rise to a fiber functor

$$\omega_x : \text{Mic}_{uni}(X/K) \rightarrow \text{Vec}_K.$$

Pulling back  $\mathcal{P}_{dR}(X, \omega) \rightarrow S \times_K S$  via the inclusion  $S \rightarrow S \times_K S$  that sends  $s$  to  $(s, x)$  we obtain a torsor  $\mathcal{T}_{dR}(X, \omega)_x$  on  $S$  under the group scheme  $\pi_{1,dR}(X, \omega_x)$ .

2.1.2. *The de Rham fiber functor on  $X_M$ .* From now on we assume that the smooth projective model  $\overline{X}$  of  $X$  is isomorphic to  $\mathbb{P}^1$ . In this case, there is a canonical fiber functor [2, §12]:

$$\omega(dR) : \text{Mic}_{uni}(X/K) \rightarrow \text{Vec}_K$$

defined as follows.

For any  $(E, \nabla) \in \text{Mic}_{uni}(X/K)$  let  $(E_{can}, \nabla)$  denote the unique vector bundle with connection on  $\overline{X}$  that has logarithmic singularities with nilpotent residues at  $\overline{X} \setminus X$ . The pair  $(E_{can}, \nabla)$  is called the canonical extension of the unipotent vector bundle with connection  $(E, \nabla)$ . Since  $H^1(\overline{X}, \mathcal{O}) = 0$ , the bundle  $E_{can}$  is trivial [2, Proposition 12.3] and the functor  $\omega_{dR}$  defined as

$$\omega_{dR}(E, \nabla) := \Gamma(\overline{X}, E_{can})$$

is a fiber functor [2, §12.4]. For a subscheme  $Y$  of  $\overline{X}$  let

$$\omega(Y) : \text{Mic}_{uni}(X/K) \rightarrow \text{Vec}_Y$$

denote the fiber functor that sends  $(E, \nabla)$  to  $E_{can}|_Y$ . There are canonical isomorphisms

$$(2.1.1) \quad \omega_{dR} \otimes_K \mathcal{O}_Y \cong \omega(Y)$$

of fiber functors.

Let  $\mathcal{P}_{dR} := \mathcal{P}_{dR}(X, \omega(X))$ ,  $\mathcal{T}_{dR, x} := \mathcal{T}_{dR}(X, \omega(X))_x$ ,  $\overline{\mathcal{P}}_{dR} := \mathcal{P}_{dR}(X, \omega(\overline{X}))$  and  $\overline{\mathcal{T}}_{dR, x} := \mathcal{T}_{dR}(X, \omega(\overline{X}))_x$ . Finally let  $\mathcal{T}_{dR}$  and  $\overline{\mathcal{T}}_{dR}$  denote the torsors  $\mathcal{T}_{dR, x}$  and  $\overline{\mathcal{T}}_{dR, x}$  after the identification (2.1.1) of  $\omega_{dR}$  with  $\omega(x)$ . Thus they are torsors under  $\pi_{1, dR}(X) := \pi_{1, dR}(X, \omega_{dR})$  and depend only on  $X$ .

2.1.3. *Connection on the fundamental torsor.* Let  $\Delta_X$  denote the diagonal in  $X \times_K X$  and  $\Delta_X^{(1)}$  denote its first infinitesimal neighborhood. By the definition of  $\mathcal{P}_{dR}$ , the sections of its restriction to  $\Delta_X^{(1)}$  are  $\otimes$ -isomorphisms from  $p_2^{(1)*}\omega(X)$  to  $p_1^{(1)*}\omega(X)$  where  $p_i^{(1)} : \Delta_X^{(1)} \rightarrow X$  are the two projections. If  $(E, \nabla) \in \text{Mic}_{uni}(X/K)$ , then  $p_i^{(1)*}\omega(X)(E, \nabla) = p_i^{(1)*}(E)$  and the connection  $\nabla$  induces an isomorphism from  $p_2^{(1)*}(E)$  to  $p_1^{(1)*}(E)$  reducing to the identity on the diagonal. This in turn induces an isomorphism between the above fiber functors, and hence a section of  $\mathcal{P}_{dR}|_{\Delta_X^{(1)}}$  over  $\Delta_X^{(1)}$  which is the identity section when restricted to  $\Delta_X$ .

Therefore there is a canonical section of the restriction of  $\mathcal{P}_{dR}$  to  $\Delta_X^{(1)}$  which is the identity section on  $\Delta_X$ . This gives a connection on the  $\pi_{1, dR}(X)$ -torsor  $\mathcal{T}_{dR}$ . Note that because of the canonical isomorphisms (2.1.1)  $\mathcal{T}_{dR} \cong \pi_{1, dR}(X) \times_K X$  and  $\overline{\mathcal{T}}_{dR} \cong \pi_{1, dR}(X) \times_K \overline{X}$ . A connection on  $\mathcal{T}_{dR}$  is nothing other than a morphism  $\Delta_X^{(1)} \rightarrow \pi_{1, dR}(X)$  whose restriction to  $\Delta_X$  is the constant map to the identity element of  $\pi_{1, dR}(X)$ . This in turn is equivalent to giving a section of  $\text{Lie } \pi_{1, dR}(X) \otimes_K \Gamma(X, \Omega_{X/K}^1)$ . By [2, §12.12], the connection on  $\mathcal{T}_{dR}$  is the one that corresponds to the canonical section  $\alpha$  of  $H_{dR}^1(X) \otimes_K H_{dR}^1(X) \subseteq \text{Lie } \pi_{1, dR}(X) \otimes_K \Gamma(X, \Omega_{X/K}^1)$ .

From now on we let  $X = X'_M$  and  $K = \mathbb{Q}_p(\zeta)$ . The image of  $\alpha$  in  $\text{Lie } \pi_{1, dR}(X) \otimes_K \Gamma(X, \Omega_{X/K}^1)$  under the canonical maps above can be described as follows. For any  $x \in \overline{X} \setminus X$  and  $(E, \nabla)$ , we have the residue endomorphism

$$\text{res}_x : E_{can}(x) \rightarrow E_{can}(x),$$

induced by the map that sends the local section  $u$  of  $E_{can}$  near  $x$ , to  $(\nabla(u), t \frac{\partial}{\partial t})$ , where  $t$  is a uniformizer at  $x$ . The residue endomorphism is independent of the choice of a uniformizer and satisfies,

$$\text{res}_x((E_1, \nabla_1) \otimes (E_2, \nabla_2)) = 1 \otimes \text{res}_x(E_2, \nabla_2) + \text{res}_x(E_1, \nabla_1) \otimes 1.$$

Hence  $\text{res}_x \in \text{Lie } \pi_{1,dR}(X, \omega(x))$ .

Under the identification (2.1.1), for  $1 \leq i \leq M$ , we let  $e_i \in \text{Lie } \pi_{1,dR}(X'_M)$  correspond to  $\text{res}_{\zeta^i}$  and  $e_0$  to  $\text{res}_0$ . If we also put  $\omega_0 := d\log z$  and  $\omega_i := d\log(z - \zeta^i)$ , for  $1 \leq i \leq M$ , then the section of  $\text{Lie } \pi_{1,dR}(X'_M) \hat{\otimes}_K \Gamma(X, \Omega^1_{X'_M/K})$  that corresponds to the connection on  $\mathcal{T}_{dR}$  is  $\sum_{0 \leq i \leq M} e_i \omega_i$ .

The de Rham fundamental group of  $X'_M$  has a simple description. For any  $K$ -algebra  $A$ , denote the associative (non-commutative) algebra of formal power series in  $\{e_i | 0 \leq i \leq M\}$  over  $A$  by  $A\langle\langle e_0, \dots, e_M \rangle\rangle$  and let

$$\mathcal{U}_{dR}(A) := A\langle\langle e_0, \dots, e_M \rangle\rangle.$$

Then the universal enveloping algebra of  $\pi_{1,dR}(X'_M)$  is  $\mathcal{U}_{dR}(X'_M)(K)$ . The co-product of the Hopf algebra structure on  $\mathcal{U}_{dR}(A)$  is induced by the fact that  $e_i$  are primitive elements:  $\Delta(e_i) = 1 \otimes e_i + e_i \otimes 1$ , for  $1 \leq i \leq M$ . The  $A$ -valued points of  $\pi_{1,dR}(X'_M)$  then correspond to the group-like elements in  $\mathcal{U}_{dR}(A)$ , i.e. elements  $g$  satisfying  $\Delta(g) = g \hat{\otimes} g$  and with constant term equal to 1. For any  $g$  let  $\underline{g}$  denote the image of  $g$  under the Hopf algebra automorphism of  $\mathcal{U}_{dR}(A)$  that sends  $e_i$  to  $p^{-1}e_i$ , for all  $i$ .

The canonical connection on  $\mathcal{T}_{dR} = \pi_{1,dR}(X'_M) \times X'_M$  can be described as follows. A section of  $\mathcal{T}_{dR}$  over  $X'_M$  is given by a group-like element in  $\alpha(z) \in \mathcal{U}_{dR}(A'_M)$ , where  $z$  denotes the parameter on  $\text{Spec } A'_M = X'_M \subseteq \mathbb{A}_K^1$ . Let

$$d : \mathcal{U}_{dR}(A'_M) \rightarrow \mathcal{U}_{dR}(A'_M) \hat{\otimes}_{A'_M} \Omega^1_{A'_M/K}$$

denote the continuous differential extending the canonical differential  $A'_M \rightarrow \Omega^1_{A'_M/K}$  such that  $d(e_i) = 0$ , for  $0 \leq i \leq M$ . In other words, applying  $d$  to an element  $\alpha(z)$  amounts to applying  $d$  to each coefficient of  $\alpha(z)$ .

With this notation, the image of  $\alpha(z)$  in  $\text{Lie } \pi_{1,dR}(X'_M) \hat{\otimes}_{A'_M} \Omega^1_{A'_M/K}$  under the canonical connection  $\nabla$  on  $\mathcal{T}_{dR}$  is:

$$\nabla(\alpha(z)) = \alpha(z)^{-1} d\alpha(z) - \alpha(z)^{-1} \left( \sum_{0 \leq i \leq M} e_i \omega_i \right) \alpha(z).$$

**2.2. Crystalline fundamental group of  $X'_M$ .** We review the theory of the crystalline fundamental group as described in [2, §11] and [8, §2.4]. The comparison theorem between the crystalline and de Rham fundamental groups will give us the Frobenius map which will be central to what follows.

**2.2.1. The de Rham-crystalline comparison.** Let  $k$  be a perfect field of characteristic  $p$ , with  $W$  the ring of Witt vectors and  $K$  its field of fractions. For a smooth variety  $Y/k$ , we have  $\text{Isoc}_{uni}^\dagger(Y/W)$ , the category of unipotent overconvergent isocrystals on  $Y/W$  [8, §2.4.1], whose fundamental group at a fiber functor  $\omega$  is the crystalline fundamental group,  $\pi_{1,crys}^\dagger(Y, \omega)$ , of  $Y$ . Now suppose that  $Y$  has a smooth compactification  $\bar{Y}/k$  such that  $D := \bar{Y} \setminus Y$  is a simple normal crossings divisor in  $\bar{Y}$ ,

and let  $\overline{Y}_{\log}$  denote the canonical log structure on  $\overline{Y}$  associated to the divisor  $D$ . Shiho's theorem [7] implies that the restriction functor

$$(2.2.1) \quad \text{Isoc}_{uni}^c(\overline{Y}_{\log}/W) \rightarrow \text{Isoc}_{uni}^\dagger(Y/W),$$

from the category of unipotent convergent log isocrystals on  $\overline{Y}_{\log}$  to  $\text{Isoc}_{uni}^\dagger(Y/W)$ , is an equivalence of categories [8, Lemma 2].

This is in complete analogy with the situation over the field  $K$  of characteristic 0. If  $X/K$  is a smooth variety with a smooth compactification  $\overline{X}/K$  and with simple normal crossings divisor  $E := \overline{X} \setminus X$  in  $\overline{X}$  then the restriction

$$(2.2.2) \quad \text{Mic}_{uni}(\overline{X}_{\log}/K) \rightarrow \text{Mic}_{uni}(X/K)$$

gives an equivalence of categories [1, II.5.2].

The de Rham-crystalline comparison can be described as follows. Suppose that  $\overline{\mathcal{Z}}/W$  is a smooth, projective scheme with geometrically connected fibers and with  $\mathcal{F} \subseteq \overline{\mathcal{Z}}$  a relative simple normal crossings divisor. Let  $\mathcal{Z} := \overline{\mathcal{Z}} \setminus \mathcal{F}$ , and let  $(\overline{X}, X, E)$  and  $(\overline{Y}, Y, D)$  denote the corresponding data over the generic and special fibers respectively. The canonical functor

$$\text{Mic}_{uni}(\overline{X}_{\log}/K) \rightarrow \text{Isoc}_{uni}^c(\overline{Y}_{\log}/W)$$

is an equivalence which, when combined with (2.2.2) and (2.2.1) gives the equivalence

$$(2.2.3) \quad \text{Mic}_{uni}(X/K) \rightarrow \text{Isoc}_{uni}^\dagger(Y/W).$$

Choosing a (tangential) basepoint  $z$  on  $\mathcal{Z}$ , we get an isomorphism

$$\pi_{1,crys}^\dagger(Y, y) \xrightarrow{\sim} \pi_{1,dR}(X, x),$$

where  $x$  and  $y$  are the generic and special fibers of  $z$ .

Let  $\sigma : W \rightarrow W$  denote the lifting of the  $p$ -power Frobenius map on  $k$ , and let  $\mathcal{Z}^{(p)}$ , denote the base change of  $\mathcal{Z}/W$  via  $\sigma$  and  $X^{(p)}, Y^{(p)}$  etc. the corresponding fibers of  $\mathcal{Z}^{(p)}$ . The relative Frobenius morphism induces a  $\otimes$ -functor  $F^* : \text{Isoc}_{uni}^\dagger(Y^{(p)}/W) \rightarrow \text{Isoc}_{uni}^\dagger(Y/W)$ , and hence a map  $F_* : \pi_{1,crys}^\dagger(Y, y) \rightarrow \pi_{1,crys}^\dagger(Y^{(p)}, y^{(p)})$ . This, together with the above isomorphism, gives a morphism

$$(2.2.4) \quad F_* : \pi_{1,dR}(X, x) \rightarrow \pi_{1,dR}(X^{(p)}, x^{(p)}).$$

Similarly, for a pair of (tangential) basepoints  $z_1$  and  $z_2$  we obtain a morphism

$$(2.2.5) \quad F_* : {}_{x_2} \mathcal{P}_{dR}(X)_{x_1} \rightarrow {}_{x_2^{(p)}} \mathcal{P}_{dR}(X^{(p)})_{x_1^{(p)}}.$$

**2.2.2. Tangential basepoints in the crystalline case.** Tangential basepoints in dimension 1 are explained in detail in [2, §15] and [8, §3].

Let  $\mathcal{Z}/W$  be as above with relative dimension 1, for simplicity, and let  $z \in (\overline{\mathcal{Z}} \setminus \mathcal{Z})(W)$  with fibers  $x$  and  $y$ . Let  $T_z^\times(\overline{\mathcal{Z}})/W$  denote the tangent space of  $\overline{\mathcal{Z}}$  at  $z$  with the zero section removed. It is (non-canonically) isomorphic to  $\mathbb{G}_m/W$ . Fix  $w \in T_z^\times(\overline{\mathcal{Z}})(W)$ , with fibers  $v \in T_y^\times(\overline{Y})(k)$  and  $u \in T_x^\times(\overline{X})(K)$ . The *crystalline tangential basepoint at  $u$*  is a fiber functor

$$\omega_u : \text{Isoc}_{uni}^\dagger(Y/W) \rightarrow \text{Vec}_K.$$

Corresponding to the lifting  $\mathcal{Z}, z$  and  $w$  and the identification of  $\text{Isoc}_{uni}^\dagger(Y/W)$  with  $\text{Mic}_{uni}(X/K)$  described above this fiber functor corresponds to the fiber functor

$$\omega_v : \text{Mic}_{uni}(X/K) \rightarrow \text{Vec}_K$$

which associates to  $(E, \nabla)$  the fiber  $E_{can}(x)$  of its canonical extension at  $x$ . The effect of choosing different liftings and the frobenius action of the tangential basepoint are explained in detail in [8, §3].

**2.2.3. Cyclotomic  $p$ -adic multi-zeta values.** Let  $t_0$  denote the tangent vector 1 at 0 and  $t_i$  denote the tangent vector 1 at  $\zeta^i$ , for  $1 \leq i \leq M$ . Also for  $1 \leq i \leq M$ , let  $\bar{i}$  denote the unique integer such that  $1 \leq \bar{i} \leq M$  and  $M | (\bar{i} - pi)$ . Similarly, let  $\underline{i}$  denote the unique integer such that  $1 \leq \underline{i} \leq M$  and  $M | (i - p\underline{i})$  and let  $\underline{0} = \bar{0} = 0$ . By (2.1.1), for (tangential) basepoints  $x_i$  on  $X'_M$ , there are canonical isomorphisms between  $\omega_{x_i}$ . This gives a canonical element  ${}_{x_2}\gamma_{x_1}$  of  ${}_{x_2}\mathcal{P}_{dR}(X'_M)_{x_1}$ , which we call the *canonical de Rham path* from  $x_1$  to  $x_2$ .

For any  $1 \leq i \leq M$ , we have elements  ${}_{t_0}\gamma_{t_i} \cdot F_*({}_{t_{\bar{i}}}\gamma_{t_0}) \in \pi_{1,dR}(X'_M, t_0)(K)$ , with  $K = \mathbb{Q}_p(\zeta)$ . Identifying  $\omega_{t_0}$  with  $\omega_{dR}$  using (2.1.1), we obtain elements

$$g_i \in \pi_{1,dR}(X'_M)(K) \subseteq K\langle e_0, \dots, e_M \rangle,$$

for  $1 \leq i \leq M$ . Let  $g := g_M$ . We denote the coefficient of the monomial  $e_{i_1} \cdots e_{i_n}$  in  $g$  by  $g[e_{i_1} \cdots e_{i_n}]$  and call it a *cyclotomic  $p$ -adic multi-zeta value*.

The  $p$ -adic cyclotomic multi-zeta values completely determine the frobenius action on  $\pi_{1,dR}(X'_M, t_0) \xrightarrow{\sim} \pi_{1,dR}(X'_M)$  as follows.

First note that

$$(2.2.6) \quad F_*(e_0) = pe_0, \quad F_*(e_i) = pg_{\bar{i}}^{-1}e_i g_{\bar{i}}.$$

On the other hand, all the  $g_i$  are determined by  $g$  through functoriality. Let  $\alpha_i$  denote the automorphism of  $X'_M$  given by  $\alpha_i(z) = \zeta^i z$ . Then  $\alpha_{i*}(e_0) = e_0$  and  $\alpha_{i*}(e_j) = e_{i+j}$ , where  $i+j$  is between 1 and  $M$  computed modulo  $M$ . On the special fiber we have  $F \circ \alpha_{\underline{i}} = \alpha_i \circ F$ . By the functoriality of frobenius we have

$$(2.2.7) \quad \alpha_{i*}(g_j) = g_{i+j}.$$

**2.2.4. The differential equation satisfied by the frobenius.** Let us first recall the explicit description of the frobenius on  $\text{Mic}_{uni}(\overline{X}'_{M,\log}/K)$ , explained in detail in [8, §2.4.2]. Let  $\overline{\mathcal{P}}_M/W$  denote the formal scheme which is the completion of  $\overline{X}'_M$  along the closed fiber and let  $\mathcal{D}_M$  denote the divisor obtained by completing  $D_M$ . Let  $\{\overline{\mathcal{P}}_i\}_{1 \leq i \leq n}$  be an open cover of  $\overline{\mathcal{P}}_M$ , and  $\mathcal{F}_i : \overline{\mathcal{P}}_i \rightarrow \overline{\mathcal{P}}_i$  be a lifting of the frobenius such that  $\mathcal{F}_i^*(\mathcal{D}_M \cap \overline{\mathcal{P}}_i) = p \cdot (\mathcal{D}_M \cap \overline{\mathcal{P}}_i)$ . For a formal scheme  $\mathcal{P}/W$ , let  $\mathcal{P}_K$  denote the associated rigid analytic space over  $K$ . Now given  $(E, \nabla)$  in  $\text{Mic}_{uni}(\overline{X}'_{M,\log})$ , its pull-back via the frobenius is defined as the vector bundle with connection whose restriction to  $\overline{\mathcal{P}}_{iK}$  is given by  $\mathcal{F}_{i,K}^*(E, \nabla)|_{\overline{\mathcal{P}}_{i,K}}$ . The isomorphisms between the different pull-backs are given by using the fact that the connections converge within a  $p$ -adic disk of radius one and that the different liftings of the frobenius lie in the same disk [8, §2.4.2].

Let  $\overline{\mathcal{P}}$  denote the completion of  $X'_M \cup \{0, \infty\}$  along the closed fiber and let  $\mathcal{F}(z) = z^p$ . Then  $\mathcal{F} : \overline{\mathcal{P}} \rightarrow \overline{\mathcal{P}}$  is a lifting of frobenius that satisfies  $\mathcal{F}^*(0) = p \cdot (0)$  and  $\mathcal{F}^*(\infty) = p \cdot (\infty)$ . We identify the  $\pi_{1,dR}(X'_M, t_0)$ -torsor of paths that start at  $t_0$  (§2.1.1) with  $\overline{\mathcal{T}}_{dR}$  (§2.1.3) by using the identification of  $\omega(t_0)$  and  $\omega_{dR}$  (§2.1.2). The principal part of  $\mathcal{F}$  sends  $t_0$  to itself [8, §3.2.(ii)]. Then by the description of the frobenius map above, we obtain the following commutative diagram:

$$\begin{array}{ccc}
\overline{\mathcal{T}}_{dR}|_{\mathcal{U}_M} & \xrightarrow{\mathcal{F}_*} & \mathcal{F}^*\overline{\mathcal{T}}_{dR}|_{\mathcal{U}_M} \\
\downarrow \nabla & & \downarrow \mathcal{F}^*\nabla \\
\mathrm{Lie} \pi_1(X'_M) \hat{\otimes} \Omega_{\mathcal{U}_M}^1(\log(0)) & \xrightarrow{\mathrm{Lie} \mathcal{F}_*} & \mathrm{Lie} \pi_1(X'_M) \hat{\otimes} \Omega_{\mathcal{U}_M}^1(\log(0)),
\end{array}$$

where  $\mathcal{U}_M := \overline{\mathcal{P}}_K$ . Let  $\mathcal{A}_M$  denote the ring of rigid analytic functions on  $\mathcal{U}_M$ . Applying Frobenius to the section  $\gamma_{t_0}$  of  $\overline{\mathcal{T}}_{dR}|_{\mathcal{U}_M}$  and denoting  $\mathcal{F}_*(\gamma_{t_0}) \in \mathcal{U}_{dR}(\mathcal{A}_M)$  by  $g_{\mathcal{F}}(z)$ , we obtain the differential equation

$$(2.2.8) \quad - \sum_{0 \leq i \leq M} F_*(e_i) \omega_i = g_{\mathcal{F}}^{-1} dg_{\mathcal{F}} - g_{\mathcal{F}}^{-1} \left( \sum_{0 \leq i \leq M} e_i \mathcal{F}^* \omega_i \right) g_{\mathcal{F}}.$$

Putting  $g_0 = 1$ , and  $\underline{0} = \overline{0} = 0$ , we can rewrite this as:

$$(2.2.9) \quad dg_{\mathcal{F}} = \left( \sum_{0 \leq i \leq M} e_i \mathcal{F}^* \omega_i \right) g_{\mathcal{F}} - g_{\mathcal{F}} \left( \sum_{0 \leq i \leq M} p g_i^{-1} e_i g_i \omega_{\underline{i}} \right).$$

**Notation.** In the following, we let  $n_0 = 0$  and  $d_i := n_i - n_{i-1}$ , for  $i \geq 1$ . Suppose that  $\underline{s} = (s_1, \dots, s_k)$ , where  $s_j$  are positive integers,  $\underline{i} := (i_1, \dots, i_k)$ , where  $1 \leq i_j \leq M$ , and  $\underline{\alpha} := \{\alpha_1, \dots, \alpha_r\} \subseteq \{n_i | 1 \leq i \leq k\} \cup \{d_i | 1 \leq i \leq k\}$ . Then we let

$$S(\underline{s}; \underline{i}; \underline{\alpha})(z) := p^{\sum_i s_i} \sum \frac{z^{n_k}}{n_1^{s_1} \dots n_k^{s_k} \zeta_{\underline{i}}^{i_1 d_1 + \dots + i_k d_k}},$$

where the sum is taken over all  $0 < n_1 < \dots < n_k$ , which satisfy both of the following two properties:

- (i)  $p \nmid n_1$  and
- (ii)  $p | \alpha_i$ , for all  $1 \leq i \leq r$ .

If we take the sum over all  $0 < n_1 < \dots < n_k$  which satisfy (ii) then we denote the resulting series by  $T(\underline{s}, \underline{i}, \underline{\alpha})(z)$ . We use  $\underline{S}(\cdot)$  and  $\underline{T}(\cdot)$  to denote  $S(\cdot)$  and  $T(\cdot)$  without the  $p^{\sum_i s_i}$  factors.

For any power series  $f \in K[[z]]$ , we let  $f[w]$  denote the coefficient of  $z^w$  in  $f$ .

Let

$$F(\underline{s}; \underline{i}; \underline{\alpha})(n) := p^{\sum_i s_i} \sum \frac{\zeta_{\underline{i}}^{i_1 n_1 + \dots + i_k n_k}}{n_1^{s_1} \dots n_k^{s_k}},$$

where the sum is over all  $0 < n_1 < \dots < n_k < n$  that satisfy (i) and (ii) above. We denote the function obtained by taking the sum over  $0 < n_1 < \dots < n$  that satisfies (ii), by  $G(\underline{s}, \underline{i}, \underline{\alpha})(n)$ . Similarly, let  $\underline{F}$  and  $\underline{G}$  be the versions without the  $p$ -power factor. Clearly,

$$\underline{S}(\underline{s}; \underline{i}; \underline{\alpha})[n] = \frac{\zeta_{\underline{i}}^{-i_k n}}{n^{s_k}} \underline{F}(\underline{s}', \underline{i}')(n),$$

where  $\underline{s}' = (s_1, \dots, s_{k-1})$  and  $\underline{i}' = (i_2 - i_1, \dots, i_k - i_{k-1})$ , and there are similar relations for any  $\underline{\alpha}$  as above.

Using the definition of  $L^{(k)}$  for an  $M$ -power series function  $L$  in Example 5.0.4 (iv), we define  $F(s_1, (s_2); i, j)$  as follows. Noting that

$$F(s_1, s_2; i, j)(n) = p^{s_2} \sum_{0 < k < n} \frac{F(s_1; i)(k)}{k^{s_2}} \zeta_{\underline{i}}^{jk},$$

we put

$$F(s_1, (s_2); i, j)(n) = p^{s_2} \sum_{0 < k < n} F^{(s_2)}(s_1; i)(k) \zeta_{\underline{i}}^{jk}.$$

We define  $F(s_1, (s_2); i, j; \underline{\alpha})$  analogously, and we let  $F^{(\cdot)}(\cdot) := F(\cdot)^{(\cdot)}$ .

If we put  $\underline{i} = (i, j, k)$ , then we define  $S(a, (b), c; \underline{i})$  and  $S(a, (b), (c); \underline{i})$  as follows:

$$S(a, (b), c; \underline{i})[n] = \frac{p^c \zeta^{-kn}}{n^c} F(a, (b); \underline{i}')(n)$$

and

$$S(a, (b), (c); \underline{i})[n] = p^c \zeta^{-kn} F^{(c)}(a, (b); \underline{i}')(n).$$

We define  $S(a, b, (c); \underline{i}; \underline{\alpha})$  and  $S(a, (b), c; \underline{i}; \underline{\alpha})$  and  $S(a, (b), (c); \underline{i}; \underline{\alpha})$  with analogous identities.

When the limit exists, we let  $\mathcal{X}^{(\cdot)}(\cdot) := \lim_{N \rightarrow \infty} F^{(\cdot)}(\cdot)(q^N)$ .

### 3. RIGID ANALYTIC FUNCTIONS ON $\mathcal{U}_M$

Let  $D(a, r)$  and  $D(a, r)^\circ$  denote the closed and open disks of radius  $r$  around  $a$ . Then  $\mathcal{U}_M = \mathbb{P}_K^1 \setminus \cup_{1 \leq i \leq M} D(\zeta^i, 1)^\circ$ . We will need the following proposition, which is a generalization of (Prop. 2, [8]).

**Proposition 3.0.1.** *Let  $f$  be a rigid analytic function on  $\mathcal{U}_M$  with  $f(0) = 0$  and a power series expansion*

$$f(z) = \sum_{0 < n} a_n z^n$$

*around 0. Then the sequence of rational functions*

$$f_N(z) := \frac{1}{1 - z^{Mp^N}} \sum_{0 < n \leq Mp^N} a_n z^n$$

*converge uniformly on  $\mathcal{U}_M$  to  $f$ . The value of  $f$  at  $\infty$  is given by*

$$f(\infty) = - \lim_{N \rightarrow \infty} a_{Mp^N}.$$

*Proof.* Since  $f$  is rigid analytic on the affinoid  $\mathcal{U}_M$ , it is a uniform limit of rational functions with poles outside  $\mathcal{U}_M$  (§2.2, [5]). We may also assume, without loss of generality, that these rational functions are 0 at 0.

**Claim 3.0.2.** *If  $r(z)$  is a rational function with poles outside  $\mathcal{U}_M$ , then  $r(z)$  is a linear combination of functions of the form*

$$\frac{z^i}{(1 - az^M)^k},$$

*for some  $0 \leq k, 0 \leq i < M$ , and  $|1 - a| < 1$ .*

*Proof of the claim.* By the method of partial fractions,  $r(z)$  is a linear combination of rational functions of the form

$$(3.0.10) \quad \frac{1}{(1 - bz)^t},$$

with  $|b - \zeta^i| < 1$ , for some  $0 \leq i < M$ , and  $0 \leq t$ . Therefore, we need to prove the statement only for rational functions as in (3.0.10). Note that

$$\frac{1}{(1 - bz)^t} = \frac{p(z)}{(1 - az^M)^t} = \sum_{0 \leq i \leq M-1} z^i \frac{q_i(1 - az^M)}{(1 - az^M)^t},$$

for some polynomials  $p(z)$  and  $q_i(z)$ ,  $0 \leq i \leq M-1$  and  $a = b^M$ . Since  $|1 - a| < 1$ , and the left hand side does not have a pole at  $\infty$ , the right hand side is exactly as in the form stated in the claim. This proves the claim.  $\square$



Using the claim above, we will prove the following estimate on the coefficients of the Taylor expansion of  $f$  :

**Claim 3.0.3.** *For  $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ , let  $n|_N$  denote the unique integer such that  $0 < n|_N \leq Mp^N$ , and  $Mp^N$  divides  $n - n|_N$ . If we let  $c_N := \sup_{n \in \mathbb{N}} |a_n - a_{n|_N}|$ , then*

$$\lim_{N \rightarrow \infty} c_N = 0.$$

*Proof of the claim.* First we note that, for  $1 \leq k$ ,  $0 \leq i < M$  and  $|1 - a| < 1$ ,

$$\frac{z^i}{(1 - az^M)^k} = \sum_{0 \leq n} \binom{n+k-1}{k-1} a^n z^{i+Mn} =: \sum_{0 \leq n} a_n z^n;$$

satisfy the property in the claim. If  $n \not\equiv i \pmod{M}$  then  $a_n = 0$  and hence

$$\begin{aligned} c_N &= \sup_{\substack{n \in \mathbb{N} \\ n \equiv i \pmod{M}}} |a_n - a_{n|_N}| \leq \sup_{s, t \geq 0} |a_{i+M(t+sp^N)} - a_{i+Mt}| \\ &= \sup_{\substack{t \geq 0 \\ s \geq 1}} |q(t + sp^N)a^{sp^N} - q(t)| =: d_N, \end{aligned}$$

where  $q(t) := \binom{t+k-1}{k-1}$  is a polynomial of degree  $k-1$  in  $t$ . Let  $\alpha$  denote the maximum of the absolute value of the coefficients of  $q(t)$ , and  $\beta := |a-1| < 1$ . Since  $|a^p - 1| \leq \max(\beta/p, \beta^p)$ , choosing  $N_0$  sufficiently large  $|a^{p^{N_0}} - 1| \leq p^{-1}$ , and hence for  $N \geq N_0$ ,  $|a^{p^N} - 1| \leq p^{-(N-N_0)}$ . Then for  $N \geq N_0$ ,  $d_N \leq \alpha(p^{-N} + p^{-(N-N_0)})$ , and hence  $\lim_{N \rightarrow \infty} d_N = \lim_{N \rightarrow \infty} c_N = 0$ .

Since any rational function  $r(z)$ , whose poles are outside  $\mathcal{U}_M$ , is a linear combination of functions as above (Claim 3.0.2), the statement is true for  $r(z)$ . Note that for any power series  $g(z) := \sum_{0 \leq n} b_n z^n$ , which is convergent on  $D(0, 1)^\circ$  :

$$(3.0.11) \quad \sup_{0 \leq n} |b_n| \leq \sup_{|z| < 1} |g(z)|.$$

Let  $(r_m)$  be a sequence of rational functions which are 0 at 0, have poles outside  $\mathcal{U}_M$ , and which converge, uniformly on  $\mathcal{U}_M$ , to  $f$ . Letting

$$r_m(z) := \sum_{0 < n} a_n^{(m)} z^n,$$

and  $c_N^{(m)} := \sup_{n \in \mathbb{N}} |a_n^{(m)} - a_{n|_N}^{(m)}|$ ; we know that  $\lim_{N \rightarrow \infty} c_N^{(m)} = 0$ , for all  $m$ . By uniform convergence and (3.0.11),  $\lim_{m \rightarrow \infty} \sup_{N \in \mathbb{N}} |c_N^{(m)} - c_N| = 0$ . This implies the claim.  $\square$

Now, note that

$$f_{N+1}(z) - f_N(z) = \frac{1}{1 - z^{Mp^{N+1}}} \sum_{0 < n \leq Mp^{N+1}} (a_n - a_{n|_N}) z^n.$$

Note that  $z \in \mathcal{U}_M$  if and only if  $1 \leq |1 - z^M|$ . Letting  $0 < n \leq Mp^{N+1}$ ,

$$\left| \frac{z^n}{1 - z^{Mp^{N+1}}} \right| = |z^n| < 1,$$

if  $|z| < 1$ ; and

$$\left| \frac{z^n}{1 - z^{Mp^{N+1}}} \right| \leq \frac{1}{|(1/z^M)^{p^{N+1}} - 1|} \leq 1,$$

if  $1 \leq |z|$  and  $z \in \mathcal{U}_M$ . Therefore,

$$\sup_{z \in \mathcal{U}_M} |f_{N+1}(z) - f_N(z)| \leq c_N,$$

and we conclude, by Claim 3.0.3, that  $(f_N)$  converges uniformly to a rigid analytic function on  $\mathcal{U}_M$ . To see that this function, indeed, is  $f$ , we note that for  $|z| < 1$ ,  $|f(z) - f_N(z)| \leq c_N$ . Then again Claim 3.0.3 implies the assertion. The last assertion follows from  $f_N(\infty) = -a_{Mp^N}$ .  $\square$

**Corollary 3.0.4.** *Let  $f(z) = \sum_{0 < n} a_n z^n$  be as in Proposition 3.0.1, and  $0 < l \leq pM$  then*

$$\lim_{N \rightarrow \infty} |a_{lq^{N+1}} - a_{lq^N}| = 0.$$

*If  $\lim_{N \rightarrow \infty} lq^N a_{lq^N}$  exists then it is equal to 0.*

*Proof.* Since  $M|(q-1)$ ,  $Mp^N|(lq^{N+1} - lq^N)$  and hence  $|a_{lq^{N+1}} - a_{lq^N}| \leq 2c_N$ . Since  $\lim_{N \rightarrow \infty} c_N = 0$ , the first statement follows. Assume that  $\lim_{N \rightarrow \infty} lq^N a_{lq^N} = \alpha$ . Then

$$q\alpha = \lim_{N \rightarrow \infty} (lq^{N+1} a_{lq^N} + lq^{N+1} (a_{lq^{N+1}} - a_{lq^N})) = \lim_{N \rightarrow \infty} lq^{N+1} a_{lq^{N+1}} = \alpha.$$

Hence  $\alpha = 0$ .  $\square$

#### 4. COMPUTATION OF THE POLYLOGARITHMIC PART

In this section we determine the frobenius action on the polylogarithmic quotient of the fundamental group of  $X'_M$ .

**4.1. Computation of  $g_j[e_0^* e_i]$ .** Let  $e_\infty \in \text{Lie } \pi_{1,dR}(X'_M)$  denote the element which is obtained by  $\text{res}_\infty$ , the residue at  $\infty$ , as in §2.1.3.

Applying  $F_*$  to the identity

$$\sum_{0 \leq i \leq M} e_i + e_\infty = 0,$$

we get

$$\sum_{0 \leq i \leq M} g_i^{-1} e_i g_i = g_{\mathcal{F}}(\infty)^{-1} \left( \sum_{0 \leq i \leq M} e_i \right) g_{\mathcal{F}}(\infty).$$

Rewriting this, we obtain the fundamental identity

$$(4.1.1) \quad g_{\mathcal{F}}(\infty) \left( \sum_{0 \leq i \leq M} g_i^{-1} e_i g_i \right) = \left( \sum_{0 \leq i \leq M} e_i \right) g_{\mathcal{F}}(\infty).$$

From the equation (2.2.9), we obtain

$$dg_{\mathcal{F}}[e_0] = \mathcal{F}^* \omega_0 - p\omega_0 = 0,$$

and hence that  $g_{\mathcal{F}}[e_0] = 0$ , since  $g_{\mathcal{F}}(0) = 1$ .

Similarly, for  $1 \leq i \leq M$ ,

$$dg_{\mathcal{F}}[e_i] = \mathcal{F}^* \omega_i - p\omega_i,$$

which gives that

$$g_{\mathcal{F}}(z)[e_i] = p \sum_{\substack{1 \leq n \\ p \nmid n}} \frac{(\zeta^{-i} z)^n}{n},$$

for  $z \in D(0, 1)^\circ$ . Since  $g_{\mathcal{F}}[e_i]$  is a rigid analytic function Proposition 3.0.1 implies that

$$g_{\mathcal{F}}(z)[e_i] = \lim_{N \rightarrow \infty} \frac{p}{1 - z^{Mp^N}} \sum_{\substack{0 < n \leq Mp^N \\ p \nmid n}} \frac{(\zeta^{-i}z)^n}{n},$$

for  $z \in \mathcal{U}_M$ , and

$$(4.1.2) \quad g_{\mathcal{F}}(\infty)[e_i] = 0.$$

Comparing the coefficients of  $e_i e_0$  in both sides of (4.1.1) gives

$$g_{\mathcal{F}}(\infty)[e_i] + g_i[e_0] = g_{\mathcal{F}}(\infty)[e_0].$$

Using (4.1.2), this gives  $g_i[e_0] = 0$ . That  $g_i$  is group-like implies that

$$(4.1.3) \quad g_i[e_0^n] = 0,$$

for all  $0 < n$ , and  $0 \leq i \leq M$ .

Using this and the equation (2.2.9) we see, by induction, that

$$(4.1.4) \quad g_{\mathcal{F}}(z)[e_0^{s-1}e_i] = S(s; i)(z),$$

for  $z \in D(0, 1)^\circ$ . Note that if  $\alpha$  is a group-like element with  $\alpha[e_0] = 0$  then

$$(4.1.5) \quad \alpha[e_0^a e_i e_0^b] = (-1)^b \binom{a+b}{a} \alpha[e_0^{a+b} e_i].$$

This gives

$$g_{\mathcal{F}}(z)[e_0^a e_i e_0^b] = (-1)^b \binom{a+b}{a} S(a+b+1; i).$$

Using Proposition 3.0.1 as above, we get

$$(4.1.6) \quad g_{\mathcal{F}}(\infty)[e_0^a e_i e_0^b] = 0.$$

Using (2.2.9) and (4.1.3) we obtain that  $dg_{\mathcal{F}}[e_i e_0^{s-1} e_j] =$

$$\mathcal{F}^* \omega_i g_{\mathcal{F}}[e_0^{s-1} e_j] - p(g_i[e_0^{s-1} e_j] \omega_{\underline{i}} + g_{\mathcal{F}}[e_i e_0^{s-1}] \omega_{\underline{j}} + g_j^{-1}[e_i e_0^{s-1}] \omega_{\underline{j}}).$$

From (4.1.6) and the fact that the above differential is regular at  $\infty$ , we get

$$g_j^{-1}[e_i e_0^{s-1}] = -g_i[e_0^{s-1} e_j].$$

Using this and solving the differential equation we obtain that

$$g_{\mathcal{F}}(z)[e_i e_0^{s-1} e_j] = -S(s, 1; j, i; d_2) + (-1)^{s-1} S(s, 1; i, j) + g_i[e_0^{s-1} e_j](T(1; i) - T(1; j)).$$

This gives that

$$g_{\mathcal{F}}(\infty)[e_i e_0^{s-1} e_j] = (-1)^s p^{s+1} \lim_{N \rightarrow \infty} \frac{1}{Mp^N} \sum_{\substack{0 < n \leq Mp^N \\ p \nmid n}} \frac{\zeta^{(j-i)n}}{n^s}.$$

Using this we find a formula for  $g_j[e_0^s e_i]$ , with  $s \geq 1$ , as follows. First upon comparing the coefficients of  $e_0 e_i e_0^{s-1} e_j$  in (4.1.1) and using (4.1.3) and (4.1.6) we find that

$$g_{\mathcal{F}}(\infty)[e_i e_0^{s-1} e_j] = g_j^{-1}[e_0 e_i e_0^{s-1}].$$

Again by (4.1.3),  $g_j^{-1}[e_0 e_i e_0^{s-1}] = -g_j[e_0 e_i e_0^{s-1}]$  and by (4.1.5),  $g_j[e_0 e_i e_0^{s-1}] = (-1)^{s-1} s g_j[e_0^s e_i]$ . Combining these we get the following expression.

**Proposition 4.1.1.** *For  $s \geq 1$ ,*

$$g_j[e_0^s e_i] = \frac{p^{s+1}}{s} \lim_{N \rightarrow \infty} \frac{1}{Mp^N} \sum_{\substack{0 < n < Mp^N \\ p \nmid n}} \frac{\zeta^{(j-i)n}}{n^s}.$$

**4.2. An alternative expression for  $g_j[e_0^{s-1} e_i]$  when  $i \neq j$ .** First note that by the expression for  $g_{\mathcal{F}}[e_i e_0^{s-1} e_j]$  in §4.1

$$q^N g_{\mathcal{F}}[e_i e_0^{s-1} e_j][q^N] = p((-1)^{s-1} \zeta^{-i} F(s; j-i)(q^N) + g_i[e_0^{s-1} e_j](\zeta^{-i} - \zeta^{-j}))$$

**Claim 4.2.1.**  $\lim_{N \rightarrow \infty} q^N g_{\mathcal{F}}[e_i e_0^{s-1} e_j][q^N] = 0$

*Proof.* By Corollary 3.0.4 it is enough to show that the above limit exists. This follows from the observation that

$$\sum_{\substack{0 < n < q^{N+1} \\ p \nmid n}} \frac{\zeta^n}{n^s} - \sum_{\substack{0 < n < q^N \\ p \nmid n}} \frac{\zeta^n}{n^s} = \sum_{1 \leq t \leq q-1} \sum_{\substack{0 < n < q^N \\ p \nmid n}} \frac{\zeta^{n+tq^N}}{(n+tq^N)^s}$$

is congruent modulo  $q^N$  to

$$\sum_{1 \leq t \leq q-1} \zeta^t \cdot \sum_{\substack{0 < n < q^N \\ p \nmid n}} \frac{\zeta^n}{n^s} = 0.$$

□

Then the above claim gives the following:

**Proposition 4.2.2.** *For  $i \neq j$ , we have*

$$g_i[e_0^{s-1} e_j] = \frac{(-1)^{s-1}}{1 - \zeta^{j-i}} \mathcal{X}(s; j-i).$$

**4.3. Computation of  $g_j[e_i]$ .** By (2.2.7),  $g_j[e_i] = (\alpha_{j*}(g))[e_i] = g[e_{i-j}]$ . Let  $\iota : X'_M \rightarrow X'_1$  denote the inclusion. If  $i = j$  then using the functoriality of frobenius with respect to  $\iota$  we see that  $g[e_{i-j}]$  computed on  $X'_M$  is equal to  $g[e_1]$  computed on  $X'_1$ . But this last expression is 0 by [8, §5.6]. Suppose now that  $i \neq j$ . Then  $g_j[e_i] = \alpha_{i*}(g_{j-i})[e_i] = g_{j-i}[e_M]$ . Then as above, by the functoriality of frobenius for  $\iota$ ,  $g_{j-i}[e_M]$ , which is computed on  $X'_M$ , is equal to

$$(4.3.1) \quad ({}_{t_0} \gamma_{z'} \cdot F_*(z \gamma_{t_0}))[e_1],$$

which is computed on  $X'_1$ . Here  $z = \zeta^{j-i}$  and  $z' = \zeta^{j-i}$ . Note that  $\mathcal{F}$  is good lifting of frobenius on  $\mathcal{U}_1 \subseteq X'_1$ . Since  $i \neq j$ ,  $z \in \mathcal{U}_1$ , and since  $\mathcal{F}(z) = z^p = z'$ , we see that (4.3.1) is equal to  $g_{\mathcal{F}}[\zeta^{j-i}][e_1]$ . The last expression is computed by Proposition 3.0.1 to be

$$\lim_{N \rightarrow \infty} \frac{p}{1 - \zeta^{(j-i)p^N}} \sum_{\substack{0 < n < p^N \\ p \nmid n}} \frac{\zeta^{(j-i)n}}{n}.$$

Therefore we have the following expression for  $g_j[e_i]$ .

**Proposition 4.3.1.** *If  $i = j$  then  $g_j[e_i] = 0$ . Otherwise*

$$g_j[e_i] = \log \frac{1 - \zeta^{j-i}}{(1 - \zeta^{j-i})^p}.$$

5.  $M$ -POWER SERIES FUNCTIONS

In order to compute the higher depth part of the Frobenius action we first study the type of functions that appear in these computations.

**Definition 5.0.2.** Let  $n \in \mathbb{N}$  and let  $f : \mathbb{N}_{\geq n} \rightarrow \mathbb{Q}_p[\zeta]$  be any function. We say that  $f$  is an  $M$ -power series function, if there exist power series  $p_i(x) \in \mathbb{Q}_p[\zeta][[x]]$ , which converge on  $D(0, r_i)$  for some  $r_i > |p|$ , for  $0 < i \leq pM$ , such that  $f(a) = p_i(a - i)$ , for all  $a \geq n$  and  $pM \mid (a - i)$ . We let the *absolute value of  $f$*  to be the maximum of the absolute values of the  $p_i$ .

*Remark 5.0.3.* (i) By the Weierstrass preparation theorem, the power series  $p_i$  in the above definition are unique.

(ii) Fix  $0 < l \leq pM$ , and let  $f$  be as above. Then there is a power series  $p(x) \in \mathbb{Q}_p[\zeta][[x]]$  which converges on some  $D(0, r)$  with  $r > |p|$  and

$$f(lq^N) = p(lq^N),$$

for  $N$  sufficiently large.

**Example 5.0.4.** (i) Let  $s \in \mathbb{Z}$  and  $f(k) := \zeta^{ik} k^s$ , for  $p \nmid k$  and  $f(k) = 0$  for  $p \mid k$ . Then  $f$  is an  $M$ -power series function.

(ii) Clearly the sums and products of  $M$ -power series functions are  $M$ -power series functions.

(iii) Let  $f$  be an  $M$ -power series function. For any  $0 < l \leq pM$ , with  $p \mid l$  let

$$f_l := \lim_{\substack{n \rightarrow 0 \\ pM \mid (n-l)}} f(n).$$

Let  $f^{[1]}$  be defined by

$$f^{[1]}(k) = \frac{f(k) - f_l}{k},$$

if  $p \mid k$  and  $pM \mid (k - l)$ ; and  $f^{[1]}(k) = 0$ , if  $p \nmid k$ . We then see that  $f^{[1]}$  is an  $M$ -power series function. In fact, if  $p \mid l$ , and  $p$  is a power series around 0 such that  $f(n) = p(n)$  for all  $pM \mid (n - l)$  then  $f^{[1]}(n) = q(n)$ , for all  $pM \mid (n - l)$ , where

$$q(x) = \frac{p(x) - p(0)}{x}.$$

Inductively, we let  $f^{[k+1]} := (f^{[k]})^{[1]}$ .

(iv) Using the notation as above, let  $f^1$  be defined by  $f^{(1)}(k) := f^{[1]}(k)$ , if  $p \mid k$ ; and  $f^{(1)}(k) = \frac{f(k)}{k}$ , if  $p \nmid k$ . Then  $f^{(1)}$  is also an  $M$ -power series function.

**Proposition 5.0.5.** Let  $f : \mathbb{N}_{\geq n_0} \rightarrow \mathbb{Q}_p[\zeta]$  be an  $M$ -power series function. If we define  $F : \mathbb{N}_{\geq n_0} \rightarrow \mathbb{Q}_p[\zeta]$  by

$$F(n) := \sum_{n_0 \leq k \leq n} f(k)$$

then  $F$  is also an  $M$ -power series function.

*Proof.* Note that  $f$  is uniquely extended to an  $M$ -power series function  $\tilde{f}$  which is defined on all  $\mathbb{N}$ . Then since  $\tilde{F}(n) := \sum_{1 \leq k \leq n} \tilde{f}(k) = \tilde{F}(n_0 - 1) + F(n)$  for all  $n \geq n_0$ , that  $\tilde{F}$  is an  $M$ -power series function implies the same for  $F$ . Therefore, without loss of generality, we will assume that  $n_0 = 1$ .

For  $1 \leq t \leq pM$ , let

$$F_t(n) := \sum_{\substack{1 \leq k \leq n \\ pM \mid (k-t)}} f(k) = \sum_{\substack{0 \leq \alpha \leq n-t \\ pM \mid \alpha}} p_t(\alpha),$$

for  $t \leq n$  and  $F_t(n) = 0$  otherwise.

Since  $F(n) = \sum_{1 \leq t \leq pM} F_t(n)$ , it suffices to prove that each  $F_t$  is an  $M$ -power series function. Fix  $1 \leq i, t \leq pM$  and suppose first that  $t \leq i$ . Let  $p_t(x) = \sum_{0 \leq j} a_j x^j$ . By assumption there is an  $\varepsilon > 0$  such that  $\lim_{n \rightarrow \infty} a_j p^{j(1-\varepsilon)} = 0$ .

Recall the formula for the sum of the  $j$ -th powers:

$$\sum_{1 \leq m \leq n} m^j = \frac{1}{j+1} \sum_{0 \leq k \leq j} \binom{j+1}{k} (-1)^k B_k n^{j+1-k}$$

where  $B_k$  are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{0 \leq k} \frac{B_k x^k}{k!}.$$

The Von Staudt-Clausen theorem gives the bound  $|B_k| \leq p$ .

Then for  $n \geq 0$

$$\begin{aligned} F_t(i + npM) &= \sum_{0 \leq k \leq j} \frac{a_j (pM)^j}{j+1} \binom{j+1}{k} (-1)^k B_k n^{j+1-k} \\ &= \sum_{\substack{1 \leq l \\ 0 \leq k}} \frac{a_{l+k-1} (pM)^{k-1}}{l+k} \binom{l+k}{k} (-1)^k B_k (npM)^l. \end{aligned}$$

Therefore letting  $q_t(x) = \sum_{1 \leq l} b_l x^l$ , with

$$b_l = \sum_{0 \leq k} \frac{a_{l+k-1} (pM)^{k-1}}{l+k} \binom{l+k}{k} (-1)^k B_k,$$

we have  $F_t(i + npM) = q_t(npM)$ .

Note that

$$|b_l p^{l(1-\varepsilon/2)}| \leq p^{1+\varepsilon} \max_k |a_{l+k-1}| p^{(l+k-1)(1-\varepsilon)} \frac{p^{(l/2+k)\varepsilon}}{l+k}.$$

Since  $\lim_{l \rightarrow \infty} a_{l+k-1} p^{(l+k-1)(1-\varepsilon)} = 0$  and

$$\lim_{l \rightarrow \infty} \left| \frac{p^{(l/2+k)\varepsilon}}{l+k} \right| \leq \lim_{l \rightarrow \infty} \left| \frac{p^{(l+k)\frac{\varepsilon}{2}}}{l+k} \right| = 0,$$

we see that  $\lim_{l \rightarrow \infty} b_l p^{l(1-\varepsilon/2)} = 0$ .

On the other hand if  $i < t$ , then we have

$$F_t(i + npM) = F_t(t + npM) - f(t + npM) = q_t(npM) - p_t(npM).$$

This proves that  $F_t$  is an  $M$ -power series function as desired.  $\square$

**Corollary 5.0.6.** *If  $f_1, f_2, \dots, f_k$  are  $M$ -power series functions, then the function  $G$  defined by*

$$G(n_k) := \sum_{0 \leq n_1 < n_2 < \dots < n_k} f_1(n_1) f_2(n_2) \dots f_k(n_k)$$

*is an  $M$ -power series function.*

## 6. COMPUTATION OF THE HIGHER DEPTH PART

**6.1. Computation of  $g_i[e_j e_k]$ .** The regularity of  $g_{\mathcal{F}}$  at  $\infty$  implies the regularity of  $dg_{\mathcal{F}}[e_i e_j e_k]$  at  $\infty$ . By (2.2.9), this gives

$$(6.1.1) \quad -g_{\mathcal{F}}(\infty)[e_j e_k] + g_{\mathcal{F}}(\infty)[e_i e_j] + g_k^{-1}[e_i e_j] + g_j^{-1}[e_i]g_j[e_k] + g_i[e_j e_k] = 0.$$

Since by §4.1,

$$g_{\mathcal{F}}(\infty)[e_i e_j] = -g_j[e_i] = \frac{\mathcal{X}(2; i-j)}{1 - \zeta^{i-j}}$$

for  $i \neq j$ , we have, for  $i, j, k$  all distinct,  $g_k^{-1}[e_i e_j] =$

$$(6.1.2) \quad \frac{\mathcal{X}(2; j-k)}{1 - \zeta^{j-k}} - \frac{\mathcal{X}(2; i-j)}{1 - \zeta^{i-j}} + g_j[e_i]g_j[e_k] - g_i[e_j e_k].$$

Recall that

$$g_{\mathcal{F}}[e_l e_m] = -S(1, 1; m, l; d_2) + S(1, 1; l, m) + g_l[e_m](T(1; l) - T(1; m)).$$

**6.1.1. Computation of  $g_i[e_j e_i]$ .** Using (6.1.1) with  $k = i$ , the fact that  $g_{\mathcal{F}}$  and  $g_l$  are group-like we obtain:

$$g_i[e_j e_i] = \frac{1}{2}g_j[e_i]^2 - g_{\mathcal{F}}(\infty)[e_i e_j].$$

Then Proposition 4.3.1 gives the following expression.

**Proposition 6.1.1.** *Assuming that  $i \neq j$ , we have*

$$g_i[e_j e_i] = \frac{1}{2} \left( \frac{\mathcal{X}(1; i-j)}{1 - \zeta^{i-j}} \right)^2 - \frac{\mathcal{X}(2; j-i)}{1 - \zeta^{j-i}}.$$

**6.1.2. Computation of  $g_i[e_j e_k]$ .** Using the differential equation above gives that  $g_{\mathcal{F}}[e_i e_j e_k] = S(1, 1, 1; k, j, i; d_2, d_3) - S(1, 1, 1; j, k, i; d_3) - S(1, 1, 1; j, i, k; d_2)$

$$\begin{aligned} &+ S(1, 1, 1; i, j, k) - g_j[e_k](T(1, 1; j, i; d_2) - T(1, 1; k, i; d_2)) \\ &+ g_i[e_j](T(1, 1; i, k) - T(1, 1; j, k)) - g_k[e_j]S(1, 1; i, k) + g_k^{-1}[e_i e_j]T(1, k) \\ &+ g_j[e_k]S(1, 1; i, j) - g_j[e_i]g_j[e_k]T(1; j) + g_i[e_j e_k]T(1; i). \end{aligned}$$

Let us group the terms as follows.

(i)  $q^N S(\underline{s}; \underline{i}; d_2, d_3)[q^N] = 0$ , for any  $\underline{s} := (s_1, s_2, s_3)$  and  $\underline{i} := (i_1, i_2, i_3)$ .

(ii)  $q^N \underline{T}(1; i)[q^N] = \zeta^{-\underline{i}}$

(iii)  $q^N \underline{S}(s, 1; i, j)[q^N] = \zeta^{-\underline{i}} \underline{F}(s; j-i)(q^N)$ . Hence the limit is

$$\lim_{N \rightarrow \infty} q^N \underline{S}(s, 1; i, j)[q^N] = \zeta^{-\underline{i}} \underline{\mathcal{X}}(s; j-i).$$

(iv)  $q^N \underline{S}(s_1, s_2, 1; j, i, k; d_2)[q^N] = \zeta^{-\underline{k}} \underline{F}(s_1, s_2; i-j, k-i; d_2)(q^N)$

Hence  $\lim_{N \rightarrow \infty} q^N \underline{S}(s_1, s_2, 1; j, i, k; d_2)[q^N] = \zeta^{-\underline{k}} \underline{\mathcal{X}}(s_1, s_2; i-j, k-i; d_2)$ .

(v)  $q^N (\underline{S}(1, 1, 1; j, k, i; d_3) + \underline{g}_j[e_k](\underline{T}(1, 1; j, i; d_2) - \underline{T}(1, 1; k, i; d_2)))[q^N]$

is equal to

$$\zeta^{-\underline{i}} (\underline{F}(1, 1; k-j, i-k; n_2)(q^N) + \underline{g}_j[e_k]((\underline{G}(1; i-j; n_1) - \underline{G}(1; i-k; n_1))(q^N)))$$

Let us rewrite the same expression as

$$\zeta^{-\underline{i}} \sum_{\substack{0 < n_2 < q^N \\ p | n_2}} \frac{\zeta^{n_2(\underline{i}-\underline{k})}}{n_2} \left( \underline{g}_j[e_k](\zeta^{(\underline{k}-\underline{j})n_2} - 1) + \underline{F}(1; k-j)(n_2) \right).$$

**Lemma 6.1.2.** *Let  $0 < l \leq pM$  such that  $p|l$ , and  $1 \leq t$  then*

$$\lim_{\substack{n \rightarrow 0 \\ pM|(n-l)}} F(t; i)(n) = (-1)^{t-1} g[e_0^{t-1} e_i] (1 - \zeta^{il}).$$

*Proof.* By Corollary 5.0.6,  $F(t; i)$  is an  $M$ -power series function. Therefore the limit exists and is equal to

$$\lim_{N \rightarrow \infty} F(t; i)(lq^N) = p^t \lim_{N \rightarrow \infty} \sum_{\substack{0 < n_1 < lq^N \\ p \nmid n_1}} \frac{\zeta^{in_1}}{n_1^t} = p^t \sum_{0 \leq a < l} \zeta^{ia} \lim_{N \rightarrow \infty} \sum_{\substack{0 < n < q^N \\ p \nmid n}} \frac{\zeta^{in}}{n^t},$$

which is equal to  $g[e_0^{t-1} e_i] (1 - \zeta^{il})$ , if  $0 < i < M$ , by Proposition 4.2.2. On the other hand, if  $i = M$ , then the equality follows since both sides of the expressions are 0 [8, §5.11].  $\square$

The main expression is then equal to

$$\zeta^{-i} \sum_{\substack{0 < n_2 < q^N \\ p|n_2}} \zeta^{n_2(i-k)} F^{(1)}(1; k-j)(n_2) =: \zeta^{-i} F(1, (1); k-j, i-k; n_2)(q^N).$$

By Example 5.0.4 (iii),  $F^{(1)}(1; k-j; n_1)$  is an  $M$ -power series function. Then by Proposition 5.0.5, and Remark 5.0.3 (ii), the limit exists as  $N \rightarrow \infty$  and is equal to  $\zeta^{-i} \underline{\mathcal{X}}(1, (1); k-j, i-k; n_2)$ .

(vi)  $q^N (\underline{S}(1, 1, 1; i, j, k) + \underline{g}_i(e_j)(\underline{T}(1, 1; i, k) - \underline{T}(1, 1; j, k)))[q^N]$

$$\begin{aligned} &= \zeta^{-k} \sum_{0 < n_2 < q^N} \frac{\zeta^{(k-j)n_2}}{n_2} (\underline{g}_i[e_j](\zeta^{(j-i)n_2} - 1) + \underline{F}(1; j-i)) \\ &= \zeta^{-k} \sum_{0 < n < q^N} \zeta^{(k-j)n} \underline{F}^{(1)}(1; j-i)(n) \\ &= \zeta^{-k} \underline{F}(1, (1); j-i, k-j)(q^N) \end{aligned}$$

As above the limit of the main expression as  $N \rightarrow \infty$  exists and is equal to  $\zeta^{-k} \underline{\mathcal{X}}(1, (1); j-i, k-j)$ .

Therefore we have,  $g_i[e_j e_k] = -\zeta^{i-k} g_k^{-1}[e_i e_j] + \zeta^{i-j} g_j[e_i] g_j[e_k]$

$$\begin{aligned} &-\zeta^{i-j} \underline{\mathcal{X}}(1; j-i) g_j[e_k] + \zeta^{i-k} \underline{\mathcal{X}}(1; k-i) g_k[e_j] + \zeta^{i-k} \underline{\mathcal{X}}(1, 1; i-j, k-i; d_2) \\ &+ \underline{\mathcal{X}}(1, (1); k-j, i-k; n_2) - \zeta^{i-k} \underline{\mathcal{X}}(1, (1); j-i, k-j). \end{aligned}$$

Using equation (6.1.2), we see that, for  $i, j, k$  distinct:

$$\begin{aligned} &(1 - \zeta^{i-k}) g_i[e_j e_k] = \\ &\underline{\mathcal{X}}(1, (1); k-j, i-k; n_2) - \zeta^{i-k} \underline{\mathcal{X}}(1, (1); j-i, k-j) \\ &+ \zeta^{i-k} \underline{\mathcal{X}}(1, 1; i-j, k-i; d_2) - \frac{\zeta^{i-k} \underline{\mathcal{X}}(2; j-k)}{1 - \zeta^{j-k}} + \frac{\zeta^{i-k} \underline{\mathcal{X}}(2; i-j)}{1 - \zeta^{i-j}} \\ &- \frac{\zeta^{i-j} \underline{\mathcal{X}}(1; j-i) \underline{\mathcal{X}}(1; k-j)}{1 - \zeta^{k-j}} + \frac{\zeta^{i-k} \underline{\mathcal{X}}(1; j-k) \underline{\mathcal{X}}(1; k-i)}{1 - \zeta^{j-k}} \\ &+ \frac{(\zeta^{i-j} - \zeta^{i-k}) \underline{\mathcal{X}}(1; i-j) \underline{\mathcal{X}}(1; k-j)}{(1 - \zeta^{i-j})(1 - \zeta^{k-j})}. \end{aligned}$$



**Corollary 6.1.3.** *We have  $g_{\mathcal{F}}[e_i e_j e_k] =$*

$$S(1, 1, 1; k, j, i; d_2, d_3) - S(1, (1), (1); j, k, i; d_3) - S(1, 1, (1); j, i, k; d_2) \\ + S(1, (1), (1); i, j, k) - g_k[e_j]S(1, (1); i, k) + g_j[e_k]S(1, (1); i, j),$$

where  $g_k[e_j]$  is given by Proposition 4.2.2.

**6.2. An explicit formula for  $F(t; i)$ .** The following very simple proposition will be crucial throughout the paper.

**Proposition 6.2.1.** *Let  $f$  be a rigid analytic function on  $\mathcal{U}_M$  such that*

$$df = dg + h\omega_0 + \sum_{1 \leq i \leq M} \alpha_i \omega_i,$$

with  $g(z) = \sum_{0 < n} a_n z^n$ ,  $h(z) = \sum_{0 < n} b_n z^n$ ,  $\alpha_i \in \mathbb{C}_p$ . Suppose that for  $0 < l \leq pM$ ,  $\lim_{N \rightarrow \infty} (lq^N a_{lq^N} + b_{lq^N})$  exists. Then

$$\lim_{N \rightarrow \infty} (lq^N a_{lq^N} + b_{lq^N}) = \sum_{1 \leq i \leq M} \zeta^{-il} \alpha_i.$$

*Proof.* This is an immediate consequence of Corollary 3.0.4.  $\square$

Suppose that  $t \geq 1$  and  $s \geq 2$ , the equation (2.2.9) gives that

$$dg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] = \mathcal{F}^* \omega_j g_{\mathcal{F}}[e_0^{t-1} e_k e_0^{s-1}] - pg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-2}] \omega_0 - pg_j[e_0^{t-1} e_k e_0^{s-1}] \omega_{\underline{j}}.$$

Since  $g_{\mathcal{F}}[e_0^{t-1} e_k e_0^{s-1}] = (-1)^{s-1} \binom{s+t-2}{t-1} S(s+t-1; k)$  by (4.1.4), and letting  $(x)_k := x(x+1) \cdots (x+(k-1))$ , we see that

$$\frac{(-1)^s}{(t-1)!} (s)_{t-1} dS(s+t-1, 1; k, j; d_2) = \mathcal{F}^* \omega_j g_{\mathcal{F}}[e_0^{t-1} e_k e_0^{s-1}].$$

$$\text{This gives that } dg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] = \frac{(-1)^s}{(t-1)!} (s)_{t-1} dS(s+t-1, 1; k, j; d_2) \\ - pg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-2}] \omega_0 - pg_j[e_0^{t-1} e_k e_0^{s-1}] \omega_{\underline{j}}.$$

**Proposition 6.2.2.** *If we let  $g_{\mathcal{F}}(z)[e_j e_0^{t-1} e_k e_0^{s-1}] = \sum_{0 < n} c_n z^n$ , then*

$$\lim_{N \rightarrow \infty} c_{lq^N} = \zeta^{-jl} g_j[e_0^{t-1} e_k e_0^s],$$

for any  $0 < l \leq pM$ .

*Proof.* We will prove the proposition by induction on  $s$ . Note that  $g_{\mathcal{F}}[e_j e_0^{t-1} e_k] =$

$$-S(t, 1; k, j; d_2) + (-1)^{t-1} S(t, 1; j, k) + g_j[e_0^{t-1} e_k] (T(1; j) - T(1; k)),$$

by §4.1. The coefficient of  $z^n$  in  $\underline{g}_{\mathcal{F}}[e_j e_0^{t-1} e_k]$  is  $\frac{K(n)}{n}$  where  $K(n) :=$

$$-\zeta^{-nj} \underline{F}(t; j-k; d_2)(n) + (-1)^{t-1} \zeta^{-kn} \underline{F}(t; k-j)(n) + \underline{g}_j[e_0^{t-1} e_k] (\zeta^{-jn} - \zeta^{-kn}).$$

By Lemma 6.1.2 we see that

$$\lim_{N \rightarrow \infty} F(t; k-j)(lq^N) = (-1)^{t-1} g_j[e_0^{t-1} e_k] (1 - \zeta^{(k-j)l}).$$

This implies by Example 5.0.4 (iv) that  $\frac{K(n)}{n} =$

$$K^{(1)}(n) = -\zeta^{-nj} \underline{F}(t; j-k; d_2)(n) + \zeta^{-kn} \underline{F}^{(1)}(t; k-j)(n),$$

is an  $M$ -power series function and hence by Remark 5.0.3 (ii),

$$\lim_{N \rightarrow \infty} K^{(1)}(lq^N) = \lim_{N \rightarrow \infty} \zeta^{-kn} \underline{F}^{(1)}(t; k-j)(lq^N)$$

exists, for any  $0 < l \leq pM$ .

Since  $dg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0] = tdS(t+1, 1; k, j; d_2) - pg_{\mathcal{F}}[e_j e_0^{t-1} e_k] \omega_0 - pg_j[e_0^{t-1} e_k e_0] \omega_j$ , the existence of the above limit and Proposition 6.2.1 implies that

$$\lim_{N \rightarrow \infty} K^{(1)}(lq^N) = \lim_{N \rightarrow \infty} \zeta^{-\underline{k}} \underline{F}^{(1)}(t; k-j)(lq^N) = \zeta^{-\underline{j}l} \underline{g}_j[e_0^{t-1} e_k e_0].$$

This gives that  $g_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0] =$

$$\frac{1}{(t-1)!} \sum_{0 \leq r \leq 1} (r+1)_{t-1} S(t+r, 2-r; k, j; d_2) + (-1)^t S(t, (2); j, k),$$

and the coefficient of  $z^n$  in this expression is

$$\frac{\zeta^{-nj}}{(t-1)!} \sum_{0 \leq r \leq 1} (r+1)_{t-1} \frac{F(t+r; j-k; d_2)(n)}{n^{2-r}} + (-1)^t \zeta^{-nk} F^{(2)}(t; k-j; n_1)(n).$$

Since  $\frac{F(t; i; d_2)}{n^k} = F^{(k)}(t; i; d_2)$ , we inductively we arrive at the following expression for  $g_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] =$

$$\frac{(-1)^s}{(t-1)!} \sum_{0 \leq r \leq s-1} (r+1)_{t-1} S(t+r, s-r; k, j; d_2) + (-1)^{s+t} S(t, (s); j, k),$$

and the coefficient of  $z^n$  in this expression is

$$\frac{(-1)^s \zeta^{-nj}}{(t-1)!} \sum_{0 \leq r \leq s-1} F^{(s-r)}(t+r; j-k; d_2)(n) + (-1)^{s+t} \zeta^{-nk} F^{(s)}(t; k-j)(n).$$

Now let  $g_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] = \sum c_n z^n$ . Since  $c_n$  is expressed in terms of the values of an  $M$ -power series function by the above expression, the limit  $\lim_{N \rightarrow \infty} c_{lq^N}$  exists. In order to find this limit, we employ Proposition 6.2.1 in the differential equation for  $dg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^s]$  and find that

$$\lim_{N \rightarrow \infty} c_{lq^N} = \zeta^{-\underline{j}l} g_j[e_0^{t-1} e_k e_0^s].$$

Using the expression above this gives  $\lim_{N \rightarrow \infty} F^{(s)}(t; k-j)(lq^N) =$

$$(-1)^{s+t} \zeta^{(\underline{k}-\underline{j})l} g_j[e_0^{t-1} e_k e_0^s] = \frac{(-1)^t}{(t-1)!} \zeta^{(\underline{k}-\underline{j})l} (s+1)_{t-1} g_j[e_0^{s+t-1} e_k].$$

These limits determine the  $M$ -power series function  $F(t; i)$  completely as

$$F(t; i)(n) = (-1)^{t-1} (g[e_0^{t-1} e_i] (1 - \zeta^{in}) - \frac{1}{(t-1)!} \sum_{1 \leq r} \zeta^{in} (r+1)_{t-1} g[e_0^{r+t-1} e_i] n^r),$$

for  $p|n$ . □

The proof of the above proposition has the following corollaries.

**Corollary 6.2.3.** *For  $t \geq 1$ ,  $p|n$  and  $M \nmid i$ , we have*

$$p^t \sum_{\substack{0 < k \leq n \\ p \nmid k}} \frac{\zeta^{ik}}{k^t} = (-1)^{t-1} (g[e_0^{t-1} e_i] (1 - \zeta^{in}) - \frac{1}{(t-1)!} \sum_{1 \leq r} \zeta^{in} (r+1)_{t-1} g[e_0^{r+t-1} e_i] n^r).$$

**Corollary 6.2.4.** *For,  $t, s \geq 1$  and  $j \neq k$ , we have  $g_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] =$*

$$\frac{(-1)^s}{(t-1)!} \sum_{0 \leq r \leq s-1} (r+1)_{t-1} S(t+r, s-r; k, j; d_2) + (-1)^{s+t} S(t, (s); j, k).$$

**6.3. Computation of  $g_i[e_j e_k e_0^s]$ .** We already made this computation for  $s = 0$  in §6.1. We will do induction on  $s$ . So we assume that  $s > 0$ .

The differential equation gives  $dg_{\mathcal{F}}[e_i e_j e_k e_0^s] = g_{\mathcal{F}}[e_j e_k e_0^s] \mathcal{F}^* \omega_i$

$$-pg_{\mathcal{F}}[e_i e_j e_k e_0^{s-1}] \omega_0 - pg_i[e_j e_k e_0^s] \omega_{\underline{i}} - pg_j^{-1}[e_i] g_j[e_k e_0^s] \omega_{\underline{j}} - pg_{\mathcal{F}}[e_i] g_j[e_k e_0^s] \omega_{\underline{j}}.$$

We will use Proposition 6.2.1 to make the computation. Using Corollary 6.2.4,  $g_{\mathcal{F}}[e_j e_k e_0] \mathcal{F}^* \omega_i = d(-\sum_{0 \leq r \leq 1} S(1+r, 2-r, 1; k, j, i; d_2, d_3) + S(1, (2), 1; j, k, i; d_3))$  and  $-g_{\mathcal{F}}[e_i] \omega_{\underline{j}} = dS(1, 1; i, j)$ .

Note that  $lq^N S(a, b, 1; k, j, i; d_2, d_3)[lq^N] = 0$ ,

$$lq^N S(a, (b), 1; j, k, i; d_3)[lq^N] = \zeta^{-\underline{i}l} F(a, (b); k-j, i-k; n_2)(lq^N)$$

and

$$lq^N S(a, 1; i, j)[lq^N] = \zeta^{-\underline{j}l} F(a; j-i)(lq^N).$$

On the other hand using Corollary 6.1.3 we obtain an expression for  $g_{\mathcal{F}}[e_i e_j e_k]$ , and noting that:

- (i)  $S(a, (b), (1); j, k, i; d_3)[lq^N] = \zeta^{-\underline{i}l} F^{(1)}(a, (b); k-j, i-k; n_2)(lq^N)$
- (ii)  $S(a, b, (1); j, i, k; d_2)[lq^N] = \zeta^{-\underline{k}l} F^{(1)}(a, b; i-j, k-i; d_2)(lq^N)$
- (iii)  $S(a, (b), (1); i, j, k)[lq^N] = \zeta^{-\underline{k}l} F^{(1)}(a, (b); j-i, k-j)(lq^N)$
- (iv)  $S(a, (1); i, k)[lq^N] = \zeta^{-\underline{k}l} F^{(1)}(a; k-i)(lq^N).$

Since the above limits exist as  $N \rightarrow \infty$  we can use Proposition 6.2.1 and obtain that the limit of the following as  $N \rightarrow \infty$  is equal to  $-g_i[e_j e_k e_0] \zeta^{-\underline{i}l} - g_j^{-1}[e_i] g_j[e_k e_0] \zeta^{-\underline{j}l}$ :

$$\begin{aligned} & \zeta^{-\underline{i}l} F(1, (2); k-j, i-k; n_2)(lq^N) + g_j[e_k e_0] \zeta^{-\underline{j}l} F(1; j-i)(lq^N) \\ & + \zeta^{-\underline{i}l} F^{(1)}(1, (1); k-j, i-k; n_2)(lq^N) + \zeta^{-\underline{k}l} F^{(1)}(1, 1; i-j, k-i; d_2)(lq^N) \\ & - \zeta^{-\underline{k}l} F^{(1)}(1, (1); j-i, k-j)(lq^N) + g_k[e_j] \zeta^{-\underline{k}l} F^{(1)}(1; k-i)(lq^N) \\ & - g_j[e_k] \zeta^{-\underline{j}l} F^{(1)}(1; j-i)(lq^N). \end{aligned}$$

This gives the following formula, with  $i, j$ , and  $k$  pairwise distinct, for  $g_i[e_j e_k e_0]$ :

$$\begin{aligned} & -\mathcal{X}(1, (2); k-j, i-k; n_2) - \frac{\zeta^{\underline{i}-\underline{j}}}{1 - \zeta^{\underline{k}-\underline{j}}} \mathcal{X}(2; k-j) \mathcal{X}(1; j-i) \\ & -\mathcal{X}^{(1)}(1, (1); k-j, i-k; n_2) - \zeta^{\underline{i}-\underline{k}} \mathcal{X}^{(1)}(1, 1; i-j, k-i; d_2) \\ & + \zeta^{\underline{i}-\underline{k}} \mathcal{X}^{(1)}(1, (1); j-i, k-j) + \frac{\zeta^{\underline{i}-\underline{j}} \mathcal{X}(1; i-j) \mathcal{X}(2; k-j)}{(1 - \zeta^{\underline{i}-\underline{j}})(1 - \zeta^{\underline{k}-\underline{j}})} \\ & - \frac{\zeta^{\underline{i}-\underline{k}} \mathcal{X}(1; j-k) \mathcal{X}^{(1)}(1; k-i)}{1 - \zeta^{\underline{j}-\underline{k}}} + \frac{\zeta^{\underline{i}-\underline{j}} \mathcal{X}(1; k-j) \mathcal{X}^{(1)}(1; j-i)}{1 - \zeta^{\underline{k}-\underline{j}}}. \end{aligned}$$

Now Proposition 6.2.1 implies that  $g_{\mathcal{F}}[e_i e_j e_k e_0] =$

$$\begin{aligned} & - \sum_{\substack{0 \leq p, q, r \leq 1 \\ p+q+r=1}} S(1+p, 1+q, 1+r; k, j, i; d_2, d_3) + S(1, (2), (1); j, k, i; d_3) \\ & + S(1, (1), (2); j, k, i; d_3) + S(1, 1, (2); j, i, k; d_2) + g_j[e_k e_0] S(1, (1); i, j) \end{aligned}$$

$$-S(1, (1), (2); i, j, k) + g_k[e_j]S(1, (2); i, k) - g_j[e_k]S(1, (2); i, j).$$

We use this information in the differential equation for  $dg_{\mathcal{F}}[e_i e_j e_k e_0^2]$  above and this gives a formula for  $g_i[e_j e_k e_0^2]$ . Inducting on  $s$ , we find the following formulas for  $g_{\mathcal{F}}[e_i e_j e_k e_0^{s-1}]$  and  $g_i[e_j e_k e_0^s]$ . Namely,  $g_{\mathcal{F}}[e_i e_j e_k e_0^{s-1}] =$

$$\begin{aligned} & (-1)^{s-1} \sum_{\substack{0 \leq p, q, r \\ p+q+r=s-1}} S(1+p, 1+q, 1+r; k, j, i; d_2, d_3) + (-1)^s g_k[e_j]S(1, (s); i, k) \\ & + (-1)^s \sum_{\substack{0 \leq p, q \\ p+q=s-1}} S(1, (1+p), (1+q); j, k, i; d_3) + (-1)^s S(1, 1, (s); j, i, k; d_2) \\ & + \sum_{0 \leq r \leq s-1} (-1)^r g_j[e_k e_0^{s-1-r}]S(1, (1+r); i, j) + (-1)^{s-1} S(1, (1), (s); i, j, k). \end{aligned}$$

Then using the differential equation for  $dg_{\mathcal{F}}[e_i e_j e_k e_0^s]$  and using Proposition 6.2.1 and noting that

- (i)  $S(a, (b); i, k)$  contributes  $\zeta^{-\underline{k}} \mathcal{X}^{(b)}(a; k-i)$
- (ii)  $S(a, (b), (c); j, k, i; d_3)$  contributes  $\zeta^{-\underline{i}} \mathcal{X}^{(c)}(a, (b); k-j, i-k; n_2)$
- (iii)  $S(a, b, (c); j, i, k; d_2)$  contributes  $\zeta^{-\underline{k}} \mathcal{X}^{(c)}(a, b; i-j, k-j; d_2)$
- (iv)  $S(a, (b); i, j)$  contributes  $\zeta^{-\underline{j}} \mathcal{X}^{(b)}(a; j-i)$
- (v)  $S(a, (b), (c); i, j, k)$  contributes  $\zeta^{-\underline{k}} \mathcal{X}^{(c)}(a, (b); j-i, k-j).$

Therefore we obtain that  $-\zeta^{-\underline{i}} g_i[e_j e_k e_0^s] - \zeta^{-\underline{j}} g_j^{-1}[e_i] g_j[e_k e_0^s] =$

$$\begin{aligned} & (-1)^{s+1} \zeta^{-\underline{k}} g_k[e_j] \mathcal{X}^{(s)}(1; k-i) + (-1)^{s+1} \zeta^{-\underline{k}} \mathcal{X}^{(s)}(1, 1; i-j, k-j; d_2) \\ & + (-1)^{s+1} \zeta^{-\underline{i}} \sum_{0 \leq r \leq s} \mathcal{X}^{(r)}(1, (s+1-r); k-j, i-k; n_2) \\ & + \zeta^{-\underline{j}} \sum_{0 \leq r \leq s} (-1)^r g_j[e_k e_0^{s-r}] \mathcal{X}^{(r)}(1; j-i) + (-1)^s \zeta^{-\underline{k}} \mathcal{X}^{(s)}(1, (1); j-i, k-j). \end{aligned}$$

This proves the following proposition.

**Proposition 6.3.1.** *Assume that  $i, j$  and  $k$  are pairwise distinct and that  $s > 0$ . Then we have  $g_i[e_j e_k e_0^s] =$*

$$\begin{aligned} & (-1)^s \zeta^{\underline{i}-\underline{k}} \frac{\mathcal{X}(1; j-k)}{1-\zeta^{\underline{j}-\underline{k}}} \mathcal{X}^{(s)}(1; k-i) + (-1)^s \zeta^{\underline{i}-\underline{k}} \mathcal{X}^{(s)}(1, 1; i-j, k-j; d_2) \\ & + (-1)^s \sum_{0 \leq r \leq s} \mathcal{X}^{(r)}(1, (s+1-r); k-j, i-k; n_2) \\ & - \frac{\zeta^{\underline{i}-\underline{j}}}{1-\zeta^{\underline{k}-\underline{j}}} \sum_{0 \leq r \leq s} (-1)^r \mathcal{X}(s-r+1; k-j) \mathcal{X}^{(r)}(1; j-i) \\ & + (-1)^{s+1} \zeta^{\underline{i}-\underline{k}} \mathcal{X}^{(s)}(1, (1); j-i, k-j) + \zeta^{\underline{i}-\underline{j}} \frac{\mathcal{X}(1, i-j)}{1-\zeta^{\underline{i}-\underline{j}}} \frac{\mathcal{X}(s+1; k-j)}{1-\zeta^{\underline{k}-\underline{j}}}. \end{aligned}$$

**6.4. Computation of  $g_i[e_j e_0^{s-1} e_j e_0^{t-1}]$ .** We know the answer if  $s = 1$  from the previous section. Let us assume that  $s > 1$ . Also first assume that  $t = 1$ .

**6.4.1. Computation of  $g_i[e_j e_0^{s-1} e_j]$ .** Assume that  $s > 1$ . The differential equation gives  $dg_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k] =$

$$g_{\mathcal{F}}[e_j e_0^{s-1} e_k] \mathcal{F}^* \omega_i - p g_i[e_j e_0^{s-1} e_k] \omega_{\underline{i}} - p g_{\mathcal{F}}[e_i] g_j[e_0^{s-1} e_k] \omega_{\underline{j}} - p g_j^{-1}[e_i] g_j[e_0^{s-1} e_k] \omega_{\underline{j}}$$

$$-pg_{\mathcal{F}}[e_i e_j e_0^{s-1}] \omega_{\underline{k}} - pg_{\mathcal{F}}[e_i] g_k^{-1} [e_j e_0^{s-1}] \omega_{\underline{k}} - pg_k^{-1} [e_i e_j e_0^{s-1}] \omega_{\underline{k}}.$$

Computing residues at  $\infty$  in the above expression, we obtain that  $g_i[e_j e_0^{s-1} e_k] =$

$$g_{\mathcal{F}}(\infty)[e_j e_0^{s-1} e_k] - g_{\mathcal{F}}(\infty)[e_i e_j e_0^{s-1}] - g_j^{-1}[e_i] g_j[e_0^{s-1} e_k] - g_k^{-1}[e_i e_j e_0^{s-1}]$$

Computing the residues in the differential equation for  $dg_{\mathcal{F}}[e_i e_j e_0^s]$  we obtain that

$$g_{\mathcal{F}}(\infty)[e_i e_j e_0^{s-1}] = g_{\mathcal{F}}(\infty)[e_j e_0^s] - g_i[e_j e_0^s] = (-1)^{s+1} g_i[e_0^s e_j].$$

Using the expression  $g_{\mathcal{F}}[e_j e_0^{s-1} e_k] = (-1)^s s g_k[e_0^s e_j]$  that we found in §4.1, we obtain that  $g_i[e_j e_0^{s-1} e_k] =$

$$g_k[e_i e_j e_0^{s-1}] + (-1)^s (s g_k[e_0^s e_j] + g_i[e_0^s e_j] + g_k[e_i] g_k[e_0^{s-1} e_j]) + g_j[e_i] g_j[e_0^{s-1} e_k].$$

This proves the following proposition.

**Proposition 6.4.1.** *Assume that  $i, j$ , and  $k$  are pairwise distinct and that  $s > 1$ . Then  $g_i[e_j e_0^{s-1} e_k] =$*

$$g_k[e_i e_j e_0^{s-1}] + \frac{s \mathcal{X}(s+1; j-k)}{1 - \zeta^{j-k}} + \frac{\mathcal{X}(s+1; j-i)}{1 - \zeta^{j-i}} - \frac{\mathcal{X}(1; i-k)}{1 - \zeta^{i-k}} \frac{\mathcal{X}(s; j-k)}{1 - \zeta^{j-k}} \\ + (-1)^{s-1} \frac{\mathcal{X}(1; i-j) \mathcal{X}(s; k-j)}{(1 - \zeta^{i-j})(1 - \zeta^{k-j})},$$

where  $g_k[e_i e_j e_0^{s-1}]$  is given by Proposition 6.3.1.

Using the expression for  $g_{\mathcal{F}}[e_i e_0^s e_m e_0^b]$  above we see that  $dg_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k] =$

$$d(S(s, 1, 1; k, j, i; d_2, d_3)) + (-1)^s S(s, (1), 1; j, k, i; d_3) + g_j[e_0^{s-1} e_k] S(1, 1; i, j) \\ + g_k^{-1}[e_j e_0^{s-1}] S(1, 1; i, k) + (-1)^s \sum_{0 \leq r \leq s-1} S(1+r, s-r, 1; j, i, k; d_2) \\ + (-1)^{s+1} S(1, (s), 1; i, j, k) + \sum_{1 \leq i \leq M} \alpha_i \omega_i,$$

for some  $\alpha_i \in \mathbb{Q}_p[\zeta]$ . By the same arguments as above we see that the hypotheses of Proposition 6.2.1 are satisfied, and then this proposition implies the following.

**Proposition 6.4.2.** *Suppose that  $s \geq 1$ , then we have  $g_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k] =$*

$$S(s, 1, 1; k, j, i; d_2, d_3) + (-1)^s S(s, (1), (1); j, k, i; d_3) + g_j[e_0^{s-1} e_k] S(1, (1); i, j) \\ + g_k^{-1}[e_j e_0^{s-1}] S(1, (1); i, k) + (-1)^s \sum_{0 \leq r \leq s-1} S(1+r, s-r, (1); j, i, k; d_2) \\ + (-1)^{s+1} S(1, (s), (1); i, j, k).$$

**6.4.2. Computation of  $g_i[e_j e_0^{s-1} e_k e_0^{t-1}]$ .** Assume that  $s, t > 1$ . Then the differential equation gives that  $dg_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k e_0^{t-1}] =$

$$g_{\mathcal{F}}[e_j e_0^{s-1} e_k e_0^{t-1}] \mathcal{F}^* \omega_i - pg_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k e_0^{t-2}] \omega_0 - pg_i[e_j e_0^{s-1} e_k e_0^{t-1}] \omega_{\underline{i}} \\ - pg_j^{-1}[e_i] g_j[e_0^{s-1} e_k e_0^{t-1}] \omega_{\underline{j}} - pg_{\mathcal{F}}[e_i] g_j[e_0^{s-1} e_k e_0^{t-1}] \omega_{\underline{j}}.$$

We will use Proposition 6.2.1 and do induction on  $t$ , starting with the formulas we found above for  $g_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k]$  and  $g_{\mathcal{F}}[e_a e_0^l e_b e_0^m]$ . We find that  $g_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k e_0^{t-1}] =$

$$\begin{aligned} & \frac{(-1)^{t-1}}{(s-1)!} \sum_{\substack{0 \leq r, q \\ r+q \leq t-1}} (r+1)_{s-1} S(s+r, 1-q, t-(q+r); k, j, i; d_2, d_3) \\ & + \sum_{0 \leq r \leq t-1} (-1)^r g_j[e_0^{s-1} e_k e_0^{t-1-r}] S(1, (1+r); i, j) \\ & + (-1)^{t-1} g_k^{-1}[e_j e_0^{s-1}] S(1, (t); i, k) + (-1)^{s+t} S(1, (s), (t); i, j, k) \\ & + (-1)^{s+t+1} \sum_{0 \leq r \leq s-1} S(1+r, s-r, (t); j, i, k; d_2) \\ & + (-1)^{s+t+1} \sum_{1 \leq r \leq t} S(s, (t+1-r), (r); j, k, i; d_3), \end{aligned}$$

for  $s \geq 1$  and  $t \geq 1$ .

$$\begin{aligned} & \text{We also obtain that } -\zeta^{-i} g_i[e_j e_0^{s-1} e_k e_0^{t-1}] - \zeta^{-j} g_j^{-1}[e_i] g_j[e_0^{s-1} e_k e_0^{t-1}] = \\ & \zeta^{-j} \sum_{0 \leq r \leq t-2} (-1)^{r+1} g_j[e_0^{s-1} e_k e_0^{t-2-r}] \mathcal{X}^{(1+r)}(1; j-i) \\ & + \zeta^{-k} (-1)^{t-1} g_k^{-1}[e_j e_0^{s-1}] \mathcal{X}^{(t-1)}(1; k-i) + \zeta^{-k} (-1)^{s+t} \mathcal{X}^{(t-1)}(1, (s); j-i, k-j) \\ & + \zeta^{-k} (-1)^{s+t+1} \sum_{0 \leq r \leq s-1} \mathcal{X}^{(t-1)}(1+r, s-r; i-j, k-i; d_2) \\ & + \zeta^{-i} (-1)^{s+t+1} \sum_{1 \leq r \leq t-1} \mathcal{X}^{(r)}(s, (t-r); k-j, i-k; n_2) \\ & + \zeta^{-i} (-1)^{s+t+1} \mathcal{X}(s, (t); k-j, i-k; n_2) + \zeta^{-j} g_j[e_0^{s-1} e_k e_0^{t-1}] \mathcal{X}(1; j-i). \end{aligned}$$

Using the formula for the polylogarithmic part above, we obtain the following.

**Theorem 6.4.3.** *Assume that  $s, t \geq 2$ . Then  $g_i[e_j e_0^{s-1} e_k e_0^{t-1}] =$*

$$\begin{aligned} & (-1)^{s-1} \zeta^{i-j} \binom{s+t-2}{s-1} \frac{\mathcal{X}(1; i-j)}{1-\zeta^{i-j}} \frac{\mathcal{X}(s+t-1; k-j)}{1-\zeta^{k-j}} \\ & - \zeta^{i-j} \sum_{0 \leq r \leq t-2} (-1)^{r+s} \binom{s+t-r-3}{s-1} \frac{\mathcal{X}(s+t-r-2; k-j)}{1-\zeta^{k-j}} \mathcal{X}^{(1+r)}(1; j-i) \\ & - \zeta^{i-k} ((-1)^t \frac{\mathcal{X}(s; j-k)}{1-\zeta^{j-k}} \mathcal{X}^{(t-1)}(1; k-i) + (-1)^{s+t} \mathcal{X}^{(t-1)}(1, (s); j-i, k-j) \\ & + (-1)^{s+t+1} \sum_{0 \leq r \leq s-1} \mathcal{X}^{(t-1)}(1+r, s-r; i-j, k-i; d_2)) \\ & + (-1)^{s+t} \left( \sum_{1 \leq r \leq t-1} \mathcal{X}^{(r)}(s, (t-r); k-j, i-k; n_2) + \mathcal{X}(s, (t); k-j, i-k; n_2) \right) \\ & + (-1)^s \binom{s+t-2}{s-1} \zeta^{i-j} \frac{\mathcal{X}(s+t-1; k-j)}{1-\zeta^{k-j}} \mathcal{X}(1; j-i), \end{aligned}$$

for distinct  $i, j, k$ .

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